TRANSACTIONS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 171, September 1972

BANACH SPACE BIFURCATION THEORY

BY

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ABSTRACT. We consider the bifurcation problem for the nonlinear operator equation $x = \lambda Lx + T(\lambda, x, y)$ in a real Banach space X. Here λ_0 is an eigenvalue of the bounded linear operator L, $X = N(I - \lambda_0 L) \oplus R(I - \lambda_0 L)$, $T \in C^1$ and T is of higher order in x.

New techniques are developed to simplify the solution of the bifurcation problem. When λ_0 is a simple eigenvalue, λ_0 is shown to be a bifurcation point of the homogeneous equation (i.e. $y \equiv 0$) with respect to 0. All solutions near $(\lambda_0, 0)$ are shown to be of the form $(\lambda(\epsilon), x(\epsilon))$, $0 \leq |\epsilon| < \epsilon_0$, $\lambda(\epsilon)$ and $x(\epsilon)$ are continuous and $\lambda(\epsilon)$ and $x(\epsilon)$ are in C^n or real analytic as T is in C^{n+1} or is real analytic. When T is real analytic and $\lambda(\epsilon) \not\equiv \lambda_0$ then there are at most two solution branches, and each branch is an analytic function of λ for $\lambda \neq \lambda_0$. If T is odd and analytic, for each $\lambda \in (\lambda_0 - \delta, \lambda_0)$ (or $\lambda \in (\lambda_0, \lambda_0 + \delta)$) there exist two nontrivial solutions near 0 and there are no solutions near 0 for $\lambda \in (\lambda_0, \lambda_0 + \delta)$ (or $\lambda \in (\lambda_0 - \delta, \lambda_0)$).

We then demonstrate that in each sufficiently small neighborhood of a solution of the homogeneous bifurcation problem there are solutions of the nonhomogeneous equation (i.e. $y \not\equiv 0$) depending continuously on a real parameter and on y.

If λ_0 is an eigenvalue of odd multiplicity we prove it is a point of bifurcation of the homogeneous equation.

With a strong restriction on the projection of T onto the null space of $I - \lambda_0 L$ we show λ_0 is a bifurcation point of the homogeneous equation when λ_0 is a double eigenvalue.

Counterexamples to some of our results are given when the hypotheses are weakened.

Introduction. In this paper we investigate the bifurcation problem for a nonlinear operator equation in a real Banach space. We seek solutions (λ, x, y) near $(\lambda_0, 0, 0)$ of the equation

$$(I - \lambda L)x - T(\lambda, x, y) = 0$$

where L is a bounded linear operator and T is continuously differentiable and of higher order in x, and λ_0 is an eigenvalue of L. When the null space of $I - \lambda_0 L$ is finite-dimensional we assume $I - \lambda_0 L$ is a Fredholm operator of index 0. $I - \lambda_0 L$ satisfies a comparable hypothesis when the null space is infinite-dimensional.

Krasnosel'skiĭ [5] considered the homogeneous equation (where $y \equiv 0$) for λ_0

Received by the editors May 3, 1971 and, in revised form, December 8, 1971.

AMS 1970 subject classifications. Primary 47H15; Secondary 46N05.

Key words and phrases. Banach space, bifurcation, homogeneous equation, nonhomogeneous equation, Fredholm operator, index zero, nonlinear operator, Fréchet differentiable, real analytic, symmetric n-linear form, simple eigenvalue, eigenvalue of odd multiplicity, implicit function theorem.

a simple eigenvalue. He assumed that T depends linearly on λ, and made further assumptions regarding the projection of $T(x)$ onto the null space of $I - \lambda_0 L$. With these restrictions he was able to prove that λ_0 is a bifurcation point of the equation with respect to 0.

We can dispense with these additional hypotheses and yet obtain the same conclusion for the homogeneous case when λ_0 is a simple eigenvalue.

Using simple methods based on the implicit function theorem, we develop the complete solution of the bifurcation problem when λ_0 is a simple eigenvalue. These same techniques also enable us to prove that if λ_0 is an eigenvalue of odd multiplicity then λ_0 is a bifurcation point of the homogeneous equation.

In Part I we introduce several simplifications of the bifurcation problem. We show that the problem is equivalent to finding solutions of a related operator equation in the null space of $I - \lambda_0 L$. For the homogeneous case we prove the existence of a unique function, $\hat{\lambda}(u)$, expressing λ as a function of the vectors in the null space. The function $\hat{\lambda}(u)$ has the property that if $(\lambda, u, 0)$ is a solution of the related equation then $\lambda = \hat{\lambda}(u)$.

The solution of the bifurcation problem when λ_0 is a simple eigenvalue is developed in Part II. The solutions $(\lambda, x, 0)$ of the homogeneous equation are shown to be a continuous function of a real parameter, varying in a neighborhood of 0. This function is n times continuously differentiable or real analytic as T is $n+1$ times continuously differentiable or real analytic. When T is real analytic and λ_0 is not the only eigenvalue of the nonlinear equation, there exists at most two distinct solution branches depending continuously on λ, and each branch is analytic in a deleted neighborhood of λ_0.

We proceed from the homogeneous case to the nonhomogeneous case. We demonstrate that in each sufficiently small neighborhood of a solution of the homogeneous equation there are solutions of the nonhomogeneous equation depending continuously on a real parameter and on y.

Our methods do not yield any results for the nonhomogeneous equation when $\lambda \equiv \lambda_0$. For this important case the reader is referred to the survey article of Vainberg and Trenogin [7] and the references cited therein.

When λ_0 is an eigenvalue of higher multiplicity the bifurcation problem for the homogeneous equation becomes more difficult, and in general λ_0 is not a point of bifurcation. Nevertheless, some results for λ_0 of higher multiplicity have been obtained. Krasnosel'skiĭ [4] showed that if $L + T$ is compact and is a gradient operator depending linearly on λ then λ_0 is a point of bifurcation. He also showed, by means of the degree of a mapping, that if $L + T$ is compact and λ_0 is an eigenvalue of odd multiplicity then λ_0 is a bifurcation point.

In Part III we show that under our hypotheses, if λ_0 is an eigenvalue of odd multiplicity then λ_0 is a point of bifurcation of the homogeneous equation with respect to 0. We thus eliminate the restrictions that $L + T$ be compact and T depend linearly on λ.

In addition, by imposing the strong restriction on T that one of the components of the projection of T onto the null space of $I - \lambda_0 L$ is of constant sign, we show that if λ_0 is a double eigenvalue then λ_0 is a bifurcation point of the homogeneous equation with respect to 0.

Part IV includes counterexamples to some of our results when the hypotheses are weakened. One of the examples shows that if λ_0 is not simple and the dimension of the null space of $I - \lambda_0 L$ is one, then λ_0 may fail to be a bifurcation point.

This paper is based on a part of the author's dissertation submitted to the Belfer Graduate School of Sciences, Yeshiva University in candidacy for the degree of Doctor of Philosophy. The author expresses his thanks to Professor M. S. Berger for constant encouragement and numerous helpful suggestions.

PART I. THE BIFURCATION PROBLEM AND ITS REDUCTION

1.1. In this part we introduce the general bifurcation problem and develop techniques which facilitate its solution.

Let X and Y be real Banach spaces, L a bounded linear map of X into X and let λ_0 be an eigenvalue of L. Suppose we perturb L by adding a nonlinear operator T which is continuous in a neighborhood of $(\lambda_0, 0, 0)$ in $\mathbf{R} \times X \times Y$ with range in X. We seek solutions (λ, x, y) of the equation

$$(I - \lambda L)x - T(\lambda, x, y) = 0 \tag{1.1}$$

for (λ, x, y) near $(\lambda_0, 0, 0)$.

We make the following assumptions about L and T:

H1. $R(I - \lambda_0 L)$ is closed and $X = N(I - \lambda_0 L) \oplus R(I - \lambda_0 L)$, where $R(I - \lambda_0 L)$ denotes the range of $I - \lambda_0 L$ and $N(I - \lambda_0 L)$ its kernel.

H2. $T(\lambda, 0, 0) = 0$ for all λ near λ_0. T is continuously Fréchet differentiable in a neighborhood of $(\lambda_0, 0, 0)$ and $D_2 T(\lambda, 0, 0) \equiv 0$ for all λ near λ_0.

When $N(I - \lambda_0 L)$ is a finite-dimensional subspace of X then H1 is equivalent to the assumption $N(I - \lambda_0 L) \cap R(I - \lambda_0 L) = \{0\}$ and $I - \lambda_0 L$ is a Fredholm operator of index 0 (see appendix for the proof).

There are many instances when condition H1 is met. H1 is satisfied when, for example, L is compact and $N(I - \lambda_0 L) \cap R(I - \lambda_0 L) = \{0\}$; or if X is a Hilbert space and L is a compact selfadjoint map.

In a later section it will be convenient to assume T is real analytic. For symmetric n-linear forms [6, p. 179], [7], we set

$$a_n(\underbrace{x_1, \cdots, x_1}_{n_1 \text{ times}}, \underbrace{x_2, \cdots, x_2}_{n_2 \text{ times}}, \cdots, \underbrace{x_k, \cdots, x_k}_{n_k \text{ times}}) = a_n(x_1^{n_1}, x_2^{n_2}, \cdots, x_k^{n_k}).$$

We then say an operator T is real analytic at a point x_0 if it is infinitely differentiable at x_0 and, in some neighborhood of x_0, is expressible as a Taylor series

$$T(x) = \sum_{n=0}^{\infty} a_n(x_0; (x - x_0)^n)$$

where each a_n is a continuous symmetric n-linear form. An operator T is called real analytic if it is real analytic at each point of its domain.

We shall make essential use of the following form of the implicit function theorem.

Theorem 1.1 (implicit function theorem). *Let X, Y, and Z be Banach spaces, $(x_0, y_0) \in X \times Y$ and let T be a continuous map of a neighborhood A of (x_0, y_0) into Z such that D_2T exists and is continuous in A. Suppose $T(x_0, y_0) = 0$ and $D_2T(x_0, y_0)$ is a linear homeomorphism of Y onto Z. Then there is an open neighborhood U of x_0 and an open neighborhood V of y_0 such that the following are true*:

(i) *There exists a unique continuous function f mapping U into V such that $T(x, f(x)) = 0$ for $x \in U$.*

(ii) *If for $x \in U$ and $y \in V$, $T(x, y) = 0$ then $f(x) = y$.*

(iii) *If $T \in C^1(A)$ then $f \in C^1(U)$ and the derivative of f is given by*

$$Df(x) = -(D_2T(x, f(x)))^{-1} \circ D_1T(x, f(x)).$$

(iv) *If $T \in C^n(A)$ then $f \in C^n(U)$.*

(v) *If T is real analytic in A then f is real analytic in U.*

Proof. The proof is a straightforward application of the contraction mapping principle. (See [3, pp. 194–196], [6, pp. 265–269] and [7].) □

The problem of finding solutions of equation (1.1) can be reduced to finding solutions of an equation in $N(I - \lambda_0 L)$. If $\dim N(I - \lambda_0 L) < \infty$ the original problem will thus be simplified to a finite-dimensional one. We now proceed with this reduction.

By H1, if $x \in X$, $x = u + v$ where $u \in N(I - \lambda_0 L)$ and $v \in R(I - \lambda_0 L)$ and there exist canonical projections P and P^* of X onto $N(I - \lambda_0 L)$ and X onto $R(I - \lambda_0 L)$ respectively, such that $P(x) = P(u + v) = u$ and $P^*(x) = P^*(u + v) = v$.

Since the operators $I - \lambda L$ and $I - \lambda_0 L$ commute, $R(I - \lambda_0 L)$ and $N(I - \lambda_0 L)$ are invariant under $I - \lambda L$. Thus we see that for $u \in N(I - \lambda_0 L)$ and $v \in R(I - \lambda_0 L)$, $P(I - \lambda L)(u + v) = (I - \lambda L)u$ and $P^*(I - \lambda L)(u + v) = (I - \lambda L)v$. Therefore, $x = u + v$ is a solution of equation (1.1) for λ and y if and only if for λ and y, u and v are a solution of the system of equations

$$(I - \lambda L)u - PT(\lambda, u + v, y) = 0, \tag{1.2}$$

$$(I - \lambda L)v - P^*T(\lambda, u + v, y) = 0. \tag{1.3}$$

Using the implicit function theorem we show that for each (λ, u, y) near $(\lambda_0, 0, 0)$ equation (1.3) always has a unique solution v in $R(I - \lambda_0 L)$. For convience we denote $N(I - \lambda_0 L)$ by $\mathfrak{N}$.

Theorem 1.2. *There exist a neighborhood A of $(\lambda_0, 0, 0)$ in the product space $\mathbf{R} \times \mathfrak{N} \times Y$, a neighborhood B of 0 in $R(I - \lambda_0 L)$, and a unique continuous map f of A into B, such that the following are true:*

(i) $(I - \lambda L)f(\lambda, u, y) - P^*T(\lambda, u + f(\lambda, u, y), y) = 0$ *for all* $(\lambda, u, y) \in A$.

(ii) *If* $(\lambda, u, y) \in A$ *and* $v \in B$ *satisfy equation* (1.3), *then* $f(\lambda, u, y) = v$.

(iii) $f \in C^1$ *and*

$$Df(\lambda, u, y) = -(D_2F(z, f(z)))^{-1} \circ D_1F(z, f(z)) \tag{1.4}$$

where $z = (\lambda, u, y)$ *and*

$$F((\lambda, u, y), v) = (I - \lambda L)v - P^*T(\lambda, u + v, y). \tag{1.5}$$

(iv) *If* $T \in C^n$ *then* $f \in C^n$.

(v) *If* T *is real analytic then* f *is real analytic.*

Proof. By H1, $I - \lambda_0 L$ is a 1-1 map of $R(I - \lambda_0 L)$ onto $R(I - \lambda_0 L)$. Then by the bounded inverse theorem, $I - \lambda_0 L$ is also a linear homeomorphism. The operator F defined by equation (1.5) is a continuous map of an open neighborhood of $((\lambda_0, 0, 0), 0) \in (\mathbf{R} \times \mathfrak{N} \times Y) \times R(I - \lambda_0 L)$ into $R(I - \lambda_0 L)$. $F((\lambda_0, 0, 0), 0) = 0$, D_2F is continuous and $D_2F((\lambda_0, 0, 0), 0) = I - \lambda_0 L$. The results then follow immediately from the implicit function theorem. □

Corollary 1.3. *If* (λ, u, y) *near* $(\lambda_0, 0, 0)$ *in* $\mathbf{R} \times N(I - \lambda_0 L) \times Y$ *satisfies the equation*

$$(I - \lambda L)u - PT(\lambda, u + f(\lambda, y), y) = 0 \tag{1.6}$$

then $(\lambda, u + f(\lambda, u, y), y)$ *is a solution of equation* (1.1). *Conversely, if* (λ, x, y) *near* $(\lambda_0, 0, 0)$ *satisfies equation* (1.1), *then, if* $x = u + v$, (λ, u, y) *is a solution of equation* (1.6) *and* $v = f(\lambda, u, y)$.

Proof. The proof of this corollary follows by Theorem 1.2 and our previous discussion. □

Corollary 1.3 implies that the solutions of small norm of equation (1.1) for λ and y near λ_0 and 0 are completely determined by the solutions of small norm of equation (1.6).

Definition. Let $(\lambda_0, 0)$ be a point in $\mathbf{R} \times X$ and K a continuous map of a neighborhood of $(\lambda_0, 0)$ into Y. Suppose $K(\lambda, 0) = 0$ for all λ near λ_0. The real number λ_0 is called a *bifurcation point* of the equation $K(\lambda, x) = 0$ with respect to 0 if, for any $\epsilon, \delta > 0$ there exist a real number λ and a nonzero vector $x \in X$ such that $K(\lambda, x) = 0$ and $0 \leq |\lambda - \lambda_0| < \epsilon$ and $\|x\| < \delta$.

Corollary 1.4. *λ_0 is a bifurcation point of the homogeneous equation*

$$(I - \lambda L)x - T(\lambda, x, 0) = 0 \tag{1.7}$$

with respect to 0 *if and only if λ_0 is a bifurcation point of the homogeneous equation*

$$(I - \lambda L)u - PT(\lambda, u + f(\lambda, u, 0), 0) = 0 \tag{1.8}$$

with respect to 0.

Proof. This fact follows from Corollary 1.3. □

1.2. We impose two additional assumptions on T. With these hypotheses a further simplification of our problem is possible. Let $T(\lambda, x) = T(\lambda, x, 0)$ and $f(\lambda, u) = f(\lambda, u, 0)$.

H3. Either $D_2T(\lambda, x) \to 0$ uniformly for λ near λ_0 as $x \to 0$, or $\dim N(I - \lambda_0 L) < \infty$.

H4. $\|D_1T(\lambda, x)\| = o(\|x\|)$ uniformly for λ near λ_0.

Remark. If T is either of the form $T(\lambda, x) = \lambda T(x)$ or $T(\lambda, x) = T(x)$, it is unnecessary to assume H3 and H4, since they are consequences of H2 and the definition of the derivative.

The following lemma will be used in the proof of Lemma 1.6. Its proof will be given in the appendix.

Lemma 1.5. *Let $(x_0, 0) \in X \times Y$ and T be a continuous map of a neighborhood of $(x_0, 0)$ into Z.*

(i) *If $T(x, 0) = 0$ for all x and D_2T is continuous, then as $y \to 0$, $T(x, y) \to 0$ uniformly for all x in a neighborhood of x_0.*

(ii) *If, in addition, as $y \to 0$, $D_2T(x, y) \to 0$ uniformly for x in a neighborhood of x_0, then $\|T(x, y)\| = o(\|y\|)$ uniformly for x in a neighborhood of x_0.*

With H3 and H4 we prove the following lemma:

Lemma 1.6. *There exists an interval A about λ_0 such that the following are true*:

(i) $\|f(\lambda, u)\| = o(\|u\|)$ *uniformly for* $\lambda \in A$.

(ii) $\|T(\lambda, u + f(\lambda, u))\| = o(\|u\|)$ *uniformly for* $\lambda \in A$.

(iii) $\|D_1 f(\lambda, u)\| = o(\|u\|)$ *uniformly for* $\lambda \in A$.

Proof. By Theorem 1.2, $f(\lambda, 0) = 0$ for λ near λ_0 and $D_2 f$ is continuous. Lemma 1.5 implies $f(\lambda, u) \to 0$ uniformly for λ near λ_0 as $u \to 0$. It is well known that a continuous map of a closed bounded subset of a finite-dimensional Banach space into a Banach space is uniformly continuous. Hence, by H3, $D_2T(\lambda, u + f(\lambda, u)) \to 0$ uniformly for λ near λ_0 as $u \to 0$.

By equations (1.4) and (1.5),

(1.9) $D_2 f(\lambda, u) = -[D_2 F((\lambda, u), f(\lambda, u))]^{-1} \circ [-D_2 P^* T(\lambda, u + f(\lambda, u))]$

where $F((\lambda, u), v) = (I - \lambda L)v - P^* T(\lambda, u + v)$. From equation (1.9) we get the inequality

$$\|D_2 f(\lambda, u)\| \le \|[D_2 F((\lambda, u), f(\lambda, u))]^{-1}\| \, \|D_2 P^* T(\lambda, u + f(\lambda, u))\|.$$

By continuity there is a neighborhood of $(\lambda_0, 0)$ for which $\|[D_2 F((\lambda, u), f(\lambda, u))]^{-1}\|$ is uniformly bounded. Thus there is a neighborhood A about λ_0 for which $D_2 f(\lambda, u) \to 0$ uniformly for $\lambda \in A$ as $u \to 0$. Statement (i) then follows by Lemma 1.5.

To show statement (ii) we note that the partial derivative of $T(\lambda, u + f(\lambda, u))$ with respect to u goes to 0 uniformly for $\lambda \in A$ as $u \to 0$. The result then follows from Lemma 1.5.

We now prove statement (iii). By equation (1.4)

$$D_1 f(\lambda, u) = -[D_2 F((\lambda, u), f(\lambda, u))]^{-1} \circ [D_1((I - \lambda L) f(\lambda, u)) - D_1 P^* T(\lambda, u + f(\lambda, u))].$$

A computation shows that for $\beta \in \mathbf{R}$, $D_1((I - \lambda L)v)(\beta) = -\beta L v$. Therefore,

$$\|D_1 f(\lambda, u)\| \le \|[D_2 F((\lambda, u), f(\lambda, u))]^{-1}\| (\|L\| \, \|f(\lambda, u)\| + \|D_1 P^* T(\lambda, u + f(\lambda, u))\|).$$

Statement (iii) of the lemma then follows from statement (i) and H4. □

1.3. For the remainder of this paper we assume

H5. Either X is a Hilbert space or $\dim N(I - \lambda_0 L) < \infty$.

We find it convenient to introduce an inner product on $N(I - \lambda_0 L)$. If X is a Hilbert space take $[\ ,\ ]$ to be the inner product of X. If X is not a Hilbert space choose a basis $u_1, u_2, \cdots, u_m$ of $N(I - \lambda_0 L)$ with $\|u_i\| = 1$ and let $[\ ,\ ]$ be the inner product defined by

$$[u, \hat{u}] = [a_1 u_1 + \cdots + a_m u_m, b_1 u_1 + \cdots + b_m u_m] = a_1 b_1 + \cdots + a_m b_m.$$

It can be verified that $[\ ,\ ]$ is a continuously differentiable map and there exist constants $c_1, c_2 > 0$ such that $c_1 \|u\|^2 \le [u, u] \le c_2 \|u\|^2$ for all $u \in N(I - \lambda_0 L)$ (for X a Hilbert space, $c_1 = c_2 = 1$).

Suppose (λ, u, y) is a solution of equation (1.6). Then taking inner products

$$[(I - \lambda L)u - PT(\lambda, u + f(\lambda, u, y), y), u] = 0.$$

Simplifying, we obtain

(1.10) $$\lambda = [u, u][Lu + \lambda^{-1} PT(\lambda, u + f(\lambda, u, y), y), u]^{-1}$$

when $y = 0$ this becomes

(1.11) $$\lambda = [u, u][Lu + \lambda^{-1} PT(\lambda, u + f(\lambda, u)), u]^{-1}.$$

Hence every solution of equation (1.6) or (1.8) is also a solution of equation (1.10) or (1.11). The converse of course need not be true. We can show, however, that for each u sufficiently near 0, equation (1.11) always has a unique solution λ.

Theorem 1.7. *There exist a neighborhood B of 0 in $N(I-\lambda_0 L)$, an interval A about λ_0 and a unique continuous function $\hat{\lambda}(u)$ mapping B into A such that the following are true*:

(i) *For $u \in B$, $\lambda = \hat{\lambda}(u)$ is the unique solution of equation* (1.11) *and* $\hat{\lambda}(0) = \lambda_0$.

(ii) *If $\lambda \in A$ and $u \in B$ satisfy equation* (1.11) *then* $\lambda = \hat{\lambda}(u)$.

(iii) *If $T \in C^n$ or is real analytic then in the deleted neighborhood of* 0, $\hat{\lambda}(u) \in C^n$ *or is real analytic.*

Proof. Let $G(\lambda, u) = [u, u][Lu + \lambda^{-1}PT(\lambda, u + f(\lambda, u)), u]^{-1}$. Then

$$G(\lambda, u) = [u, u]\{\lambda_0^{-1}[u, u] + \lambda^{-1}[PT(\lambda, u + f(\lambda, u)), u]\}^{-1}$$
$$= \{\lambda_0^{-1} + \lambda^{-1}[u, u]^{-1}[PT(\lambda, u + f(\lambda, u)), u]\}^{-1}.$$

By our previous discussion

$$|[u, u]^{-1}[PT(\lambda, u + f(\lambda, u)), u]| \le k_1 \|u\|^{-1} \|PT(\lambda, u + f(\lambda, u))\|.$$

Lemma 1.6 therefore, implies G is continuous in a neighborhood of $(\lambda_0, 0)$ and $G(\lambda, 0) = \lambda_0$ for all λ near λ_0. Clearly $D_1 G$ exists and is continuous in an open set not containing points of the form $(\lambda, 0)$. We will show $D_1 G$ exists and is continuous in an open neighborhood of $(\lambda_0, 0)$. For convenience let $\hat{T}(\lambda, u) = [PT(\lambda, u + f(\lambda, u)), u]$. $|\hat{T}(\lambda, u)| \le k_2 \|PT(\lambda, u + f(\lambda, u))\| \, \|u\|$ and so by Lemma 1.6

$$|\hat{T}(\lambda, u)| = o(\|u\|^2) \tag{1.12}$$

uniformly for λ near λ_0. Furthermore, the same is true for $D_1 \hat{T}$. For $\beta \in \mathbf{R}$,

$$D_1 \hat{T}(\lambda, u)(\beta)$$
$$= [D_1 PT(\lambda, u + f(\lambda, u))(\beta), u] + [D_2 PT(\lambda, u + f(\lambda, u)) \circ D_1 f(\lambda, u)(\beta), u].$$

Hence

$$|D_1 \hat{T}(\lambda, u)| \le k_2 \|u\| \{\|D_1 PT(\lambda, u + f(\lambda, u))\| + \|D_2 PT(\lambda, u + f(\lambda, u))\| \, \|D_1 f(\lambda, u)\|\}.$$

Thus by H3, H4 and Lemma 1.6

$$|D_1 \hat{T}(\lambda, u)| = o(\|u\|^2) \tag{1.13}$$

uniformly for λ near λ_0.

Since $G(\lambda, 0) = \lambda_0$ for all λ near λ_0, $D_1 G(\lambda, 0) \equiv 0$ for λ near λ_0. A computation shows that

$$D_1 G(\lambda, u) = -(G(\lambda, u))^2 [u, u]^{-1}(-\lambda^{-2}\hat{T}(\lambda, u) + \lambda^{-1} D_1 \hat{T}(\lambda, u)).$$

Therefore

$$|D_1 G(\lambda, u)| \le k_3 (G(\lambda, u))^2 \|u\|^{-2}(|\lambda|^{-2}|\hat{T}(\lambda, u)| + |\lambda|^{-1}|D_1 \hat{T}(\lambda, u)|).$$

By continuity $(G(\lambda, u))^2$ is uniformly bounded in some neighborhood of $(\lambda_0, 0)$. Therefore, by equations (1.12) and (1.13), $D_1 G(\lambda, u) \to 0$ uniformly for λ as

$u \to 0$. Thus $D_1 G$ is continuous in a neighborhood of $(\lambda_0, 0)$.

Let $F(\lambda, u) = G(\lambda, u) - \lambda$. F and $D_1 F$ are continuous. $F(\lambda_0, 0) = 0$, $D_1 F(\lambda_0, 0) = -1$ and $D_1 F$ is a linear homeomorphism for (λ, u) in a neighborhood of $(\lambda_0, 0)$. The results then follow by the implicit function theorem. (Conclusion (iii) follows when the implicit function theorem is applied to neighborhoods of points (λ, u), $u \neq 0$, near $(\lambda_0, 0)$.) $\square$

Theorem 1.7 simplifies the problem of finding solutions of equation (1.7). It is clear that if u near 0 is a solution of

$$(I - \hat{\lambda}(u)L)u - PT(\hat{\lambda}(u), u + f(\hat{\lambda}(u), u)) = 0$$

then $(\lambda, u + f(\lambda, u))$ satisfies the homogeneous equation (1.7) where $\lambda = \hat{\lambda}(u)$.

Theorem 1.8. *If for $\lambda \in \mathbf{R}$, u is a solution of sufficiently small norm of either*

$$(I - \lambda L)u - PT(\hat{\lambda}(u), u + f(\hat{\lambda}(u), u)) = 0 \tag{1.14}$$

or

$$(I - \lambda L)u - \lambda PT(u + f(\hat{\lambda}(u), u)) = 0 \tag{1.15}$$

(when $T(\lambda, x) = \lambda T(x)$) then $\hat{\lambda}(u) = \lambda$ and $(\lambda, u + f(\lambda, u))$ is a solution of equation (1.7).

Proof. We prove the theorem for equation (1.14). If (λ, u) satisfies equation (1.14) $u = \lambda Lu + PT(\beta, u + f(\beta, u))$, $\beta = \hat{\lambda}(u)$.

Since $Lu = \lambda_0^{-1}u$ there is a γ such that $\gamma u = \beta Lu + PT(\beta, u + f(\beta, u))$. Then $\gamma/\beta = [u, u]^{-1}[Lu + \beta^{-1}PT(\beta, u + f(\beta, u)), u] = \hat{\lambda}(u)^{-1} = 1/\beta$. Hence $\gamma = 1$ and the result follows. The proof for equation (1.15) is similar. $\square$

Corollary 1.9. *If for some $\beta \in \mathbf{R}$ and u in a small neighborhood of 0, $PT(\hat{\lambda}(u), u + f(\hat{\lambda}(u), u)) = \beta u$ then $(\hat{\lambda}, u + f(\hat{\lambda}, u))$ is a solution of equation* (1.7) *where $\hat{\lambda} = \hat{\lambda}(u)$.*

Proof. Since $Lu = \lambda_0^{-1}u$ there exists a β_1 such that $(I - \beta_1 L)u - PT(\hat{\lambda}, u + f(\hat{\lambda}, u)) = 0$. Hence by our previous theorem the result follows. $\square$

Theorem 1.10 (uniqueness). *Suppose (λ, x) and (λ', x') are solutions of equation* (1.7) *near $(\lambda_0, 0)$ and $Px = Px'$. Then $(\lambda, x) = (\lambda', x')$.*

Proof. By Corollary 1.3 $x = u + f(\lambda, u)$ and $x' = u + f(\lambda', u)$. As we have seen in our discussion preceding Theorem 1.7, (λ, u) and (λ', u) are solutions of equation (1.11). But by Theorem 1.7, $\lambda = \lambda'$. Hence $(\lambda, x) = (\lambda', x')$. $\square$

PART II. BIFURCATION FROM A SIMPLE EIGENVALUE

2.1. Let X and Y be Banach spaces. In this section we prove the existence of nontrivial solutions of the nonlinear operator equation

$$(I - \lambda L)x - T(\lambda, x, 0) = 0 \tag{2.1}$$

near $(\lambda_0, 0)$ in $\mathbf{R} \times X$. λ_0 is a simple eigenvalue of the bounded linear operator L, and T is a continuous map of a neighborhood of $(\lambda_0, 0, 0)$ in $\mathbf{R} \times X \times Y$ into X. We recall that an eigenvalue λ is called simple if all the solutions of the equations $(I - \lambda L)^n x = 0$ $(n = 1, 2, \cdots)$ are simultaneously solutions of the equation $(I - \lambda L)x = 0$ and $\dim N(I - \lambda L) = 1$, [5].

We assume $I - \lambda_0 L$ is a Fredholm operator of index 0 and T satisfies conditions H2–H4 of Part I. Under these hypotheses the reductions of Part I apply to our equation.

To simplify the notation we express equation (2.1) as

$$(I - \lambda L)x - T(\lambda, x) = 0. \tag{2.2}$$

For our work in this section and in §2.2 let u_0 be a fixed normalized vector in $N(I - \lambda_0 L)$.

Theorem 2.1. *Suppose λ_0 is a simple eigenvalue of L, $I - \lambda_0 L$ is a Fredholm operator of index* 0, *and T satisfies conditions* H2–H4 *of Part* I.

(i) *λ_0 is a bifurcation point of equation* (2.2) *with respect to* 0.

(ii) *There exists an interval A about* 0, *such that for each nonzero $\epsilon \in A$ equation* (2.2) *has a unique nontrivial solution of the form*

$$x = x(\epsilon) = \epsilon u_0 + v(\epsilon), \qquad \lambda = \lambda(\epsilon),$$

where $v(\epsilon) \in R(I - \lambda_0 L)$, $v(\epsilon)$ and $\lambda(\epsilon)$ are continuous, $\|v(\epsilon)\| = o(|\epsilon|)$ and $\lambda(0) = \lambda_0$.

(iii) *Moreover, every solution near $(\lambda_0, 0)$ of equation* (2.2) *is of the form given in statement* (ii).

(iv) *If $T(\lambda, x) = \lambda T(x)$ then*

$$\lambda(\epsilon) = \lambda_0\{1 + \lambda_0 \epsilon^{-1}[PT(x(\epsilon)), u_0]\}^{-1}. \tag{2.3}$$

Proof. We have shown in Part I that if $(\lambda, u) \in \mathbf{R} \times N(I - \lambda_0 L)$ is a solution of equation (1.8) then (λ, u) is also a solution of equation (1.11). When λ_0 is a simple eigenvalue the converse is also true. Suppose $(\lambda, u) = (\lambda, \epsilon u_0)$ is a solution of the equation (1.11). Then $[(I - \lambda L)u - PT(\lambda, u + f(\lambda, u)), u] = 0$. Dividing by ϵ we get

$$(I - \lambda L)u - PT(\lambda, u + f(\lambda, u)) = [(I - \lambda L)u - PT(\lambda, u + f(\lambda, u)), u_0]u_0 = 0.$$

Consequently, $(\lambda, u) = (\lambda, \epsilon u_0)$ is a solution of equation (1.8). By Theorem 1.7 there exists a neighborhood A of 0 in R such that for each $\epsilon \in A$, $(\hat{\lambda}(\epsilon u_0), \epsilon u_0)$ is the unique solution of equation (1.11). Thus, by Corollary 1.3, for each nonzero $\epsilon \in A$, $(\lambda(\epsilon), x(\epsilon))$ is the unique nontrivial solution of equation (2.2), where

$$x(\epsilon) = \epsilon u_0 + v(\epsilon) = \epsilon u_0 + f(\lambda(\epsilon), \epsilon u_0) \tag{2.4}$$

and

$$\lambda(\epsilon) = \hat{\lambda}(\epsilon u_0). \tag{2.5}$$

Since $f(\lambda, u) \in R(I - \lambda_0 L)$, $v(\epsilon) \in R(I - \lambda_0 L)$. The continuity of $x(\epsilon)$ and $\lambda(\epsilon)$ is a consequence of their definitions and of Theorems 1.2 and 1.7. Moreover, since $(\hat{\lambda}(\epsilon u_0), \epsilon u_0)$ is the unique solution of equation (1.11), $\lambda(0) = \hat{\lambda}(0) = \lambda_0$. The equation $\|v(\epsilon)\| = o(|\epsilon|)$ is a consequence of Lemma 1.6. Therefore, we can also conclude that λ_0 is a bifurcation point of equation (2.2) with respect to 0.

Statement (iii) is seen to be a consequence of the uniqueness of $f(\lambda, u)$ and $\hat{\lambda}(u)$.

To verify statement (iv), suppose $T(\lambda, x) = \lambda T(x)$. Then, by Theorem 1.7, equation (1.11) becomes

$$\hat{\lambda}(\epsilon u_0) = \lambda_0\{1 + \lambda_0 \epsilon^{-1}[PT(\epsilon u_0 + f(\hat{\lambda}(\epsilon u_0), \epsilon u_0)), u_0]\}^{-1}$$

which gives us equation (2.3). □

The functions $v(\epsilon)$ and $\lambda(\epsilon)$ of Theorem 2.1 are defined in terms of $f(\lambda, u)$ and $\hat{\lambda}(u)$. We therefore, reach several conclusions about the differentiability of $v(\epsilon)$ and $\lambda(\epsilon)$.

Theorem 2.2. *If in addition to the hypotheses of Theorem* 2.1, $T \in C^n$, *then the following are true*:

(i) $v(\epsilon) \in C^{n-1}(A)$ *and* $\lambda(\epsilon) \in C^{n-1}(A)$.

(ii) $v(\epsilon) \in C^n(A - \{0\})$ *and* $\lambda(\epsilon) \in C^n(A - \{0\})$.

(iii) *If* $n \geq 2$ *then* $|\lambda(\epsilon) - \lambda_0| = O(|\epsilon|)$.

Proof. Equation (1.11) is equivalent to

$$\lambda = \{\lambda_0^{-1} + (\lambda[u, u])^{-1}[PT(\lambda, u + f(\lambda, u)), u]\}^{-1}.$$

Hence $\lambda = (\lambda_0^{-1} + \epsilon^{-1}E(\lambda, \epsilon))^{-1}$ where $E(\lambda, \epsilon) = \lambda^{-1}[PT(\lambda, \epsilon u_0 + f(\lambda, \epsilon u_0)), u_0]$. E is a function of two real variables. $E(\lambda, 0) = 0$ for all λ near λ_0 and $E \in C^n$. By a real variables argument $\epsilon^{-1}E(\lambda, \epsilon) \in C^{n-1}$. Since $\hat{\lambda}(\epsilon u_0)$ is defined via the implicit function theorem, $\hat{\lambda}(\epsilon u_0) \in C^{n-1}$ in a neighborhood of 0. Hence by equations (2.4) and (2.5) and Theorems 1.2 and 1.7 statements (i) and (ii) follow.

If $n \geq 2$ then by statement (i), $\lambda(\epsilon) \in C^1$. Therefore, $\lambda(\epsilon)$ must satisfy a uniform Lipschitz condition near 0 and statement (iii) follows. □

Theorem 2.3. *If in addition to the hypotheses of Theorem* 2.1, T *is also real analytic then so are* $v(\epsilon)$ *and* $\lambda(\epsilon)$. *This implies that all solutions of equation* (2.2) *near* $(\lambda_0, 0)$ *are of the form*

$$x = \epsilon u_0 + \sum_{n=2}^{\infty} \epsilon^n v_n, \tag{2.6}$$

$$\lambda = \lambda_0 + \sum_{n=1}^{\infty} \lambda_n \epsilon^n. \tag{2.7}$$

Proof. Let $E(\lambda, \epsilon)$ be as defined in Theorem 2.2. Then, since E is real analytic

$$E(\lambda, \epsilon) = \sum_{n=0}^{\infty} f_n(\lambda)\epsilon^n$$

where f_n is real analytic for $n = 0, 1, 2, \cdots$. However, since $E(\lambda, 0) = 0$ and $D_2E(\lambda, 0) = 0$ for λ near λ_0, $f_0(\lambda) \equiv 0$ and $f_1(\lambda) \equiv 0$. Therefore, $\epsilon^{-1}E(\lambda, \epsilon)$ is real analytic. The function $\hat{\lambda}(\epsilon u_0)$ is defined by an application of the implicit function theorem and is therefore real analytic. Hence $\lambda(\epsilon)$ is real analytic. In addition, since the composition of analytic functions is real analytic, it follows from equation (2.4) that $v(\epsilon)$ is also real analytic. □

Many problems in mathematics require that for each λ near λ_0 one determine the number of solutions of equation (2.2). Since for each ϵ near 0 there is a unique eigenvector $x(\epsilon)$, to arrive at the number of eigenvectors corresponding to an eigenvalue λ one need only compute the number of distinct ϵ such that $\lambda = \lambda(\epsilon)$. In general each eigenvalue may have many eigenvectors (see Example 3 in Part IV). If, however, T is real analytic then unless λ_0 is the sole eigenvalue each eigenvalue has at most two eigenvectors.

Theorem 2.4. *Suppose L and T are as in Theorem 2.3.*

(i) *If $\lambda(\epsilon) \equiv \lambda_0$ then λ_0 is the only eigenvalue of equation (2.2) for small $\|x\|$.*

(ii) *If $\lambda(\epsilon) \not\equiv \lambda_0$ there is a smallest positive integer N such that the coefficient λ_N in the power series (2.7) is nonvanishing. In addition, equation (2.2) does not have nonzero solutions of small norm for $\lambda = \lambda_0$.*

(iii) *If N is odd, for each λ near λ_0, equation (2.2) has exactly one nontrivial solution $\hat{x}(\lambda)$ near 0. The function $\hat{x}(\lambda)$ is continuous for $\lambda \in (\lambda_0 - \delta_1, \lambda_0 + \delta_2)$. Moreover, if $N = 1$, $\hat{x}(\lambda)$ is real analytic in $(\lambda_0 - \delta_1, \lambda_0 + \delta_2)$, and if $N > 1$, $\hat{x}(\lambda)$ is real analytic in $(\lambda_0 - \delta_1, \lambda_0)$ and in $(\lambda_0, \lambda_0 + \delta_2)$.*

(iv) *If N is even and $\lambda_N > 0$ $(\lambda_N < 0)$, for each $\lambda \in (\lambda_0, \lambda_0 + \delta)$ $(\lambda \in (\lambda_0 - \delta, \lambda_0))$ equation (2.2) has exactly two nontrivial solutions $\hat{x}_1(\lambda)$ and $\hat{x}_2(\lambda)$ near zero and there are no solutions of small norm for $\lambda \in (\lambda_0 - \delta, \lambda_0)$ $(\lambda \in (\lambda_0, \lambda_0 + \delta))$. The functions $\hat{x}_1(\lambda)$ and $\hat{x}_2(\lambda)$ are continuous for $\lambda \in [\lambda_0, \lambda_0 + \delta)$ $(\lambda \in (\lambda_0 - \delta, \lambda_0])$. In addition, $\hat{x}_1(\lambda)$ and $\hat{x}_2(\lambda)$ are real analytic in $(\lambda_0, \lambda_0 + \delta)$ $((\lambda_0 - \delta, \lambda_0))$.*

Proof. By virtue of Theorem 2.3 and the analyticity of $\lambda(\epsilon)$, statements (i) and (ii) are obvious.

The rest of the theorem is proved by considering the derivative of $\lambda(\epsilon)$:

$$\lambda'(\epsilon) = \sum_{n=N}^{\infty} \lambda_n n\epsilon^{n-1}.$$

Therefore $\lambda'(\epsilon) = \lambda_N N\epsilon^{N-1} + h(\epsilon)$, where $|h(\epsilon)| = o(|\epsilon|^{N-1})$.

Suppose N is odd. Then $N-1$ is even and for $\epsilon \neq 0$, $\lambda_N N\epsilon^{N-1}$ has constant sign. Since $|b(\epsilon)| = o(|\epsilon|^{N-1})$, $\operatorname{sign} \lambda'(\epsilon) = \operatorname{sign} \lambda_N N\epsilon^{N-1}$ for ϵ near 0. Therefore, $\lambda(\epsilon)$ is a monotone function on an interval $(-\epsilon_0, \epsilon_0)$. The range of $\lambda(\epsilon)$ must then be an interval $(\lambda_0 - \delta_1, \lambda_0 + \delta_2)$.

If $\lambda_1 \neq 0$ then $\lambda'(\epsilon) \neq 0$ for all $\epsilon \in (-\epsilon_0, \epsilon_0)$. By the implicit function theorem $\lambda(\epsilon)$ has a real analytic inverse $i(\lambda)$. The map $\hat{x}(\lambda) = x(i(\lambda))$ for $\lambda \in (\lambda_0 - \delta_1, \lambda_0 + \delta_2)$, gives us statement (iii) for $N = 1$.

If $N > 1$ and odd, $\lambda'(\epsilon) \neq 0$ for $\epsilon \neq 0$, $\epsilon \in (-\epsilon_0, \epsilon_0)$. For each $\delta > 0$ the function $\lambda(\epsilon)$ restricted to $(-\epsilon_0, -\delta]$ has a real analytic inverse and the function $\lambda(\epsilon)$ restricted to $[\delta, \epsilon_0)$ has a real analytic inverse. Thus there exists a continuous inverse $i(\lambda)$ of $\lambda(\epsilon)$ mapping $(\lambda_0 - \delta_1, \lambda_0 + \delta_2)$ into $(-\epsilon_0, \epsilon_0)$. Furthermore, $i(\lambda)$ is real analytic in $(\lambda_0 - \delta_1, \lambda_0)$ and in $(\lambda_0, \lambda_0 + \delta_2)$. The map $\hat{x}(\lambda) = x(i(\lambda))$, $\lambda \in (\lambda_0 - \delta_1, \lambda_0 + \delta_2)$ gives us statement (iii).

Suppose N is even. We assume $\lambda_N > 0$. The case of $\lambda_N < 0$ is handled analogously. $N - 1$ is odd, which means $\lambda_N N\epsilon^{N-1} < 0$ for $\epsilon < 0$ and $\lambda_N N\epsilon^{N-1} > 0$ for $\epsilon > 0$. Hence for $\epsilon < 0$ and near 0, $\lambda'(\epsilon) < 0$ and $\lambda(\epsilon)$ is thus a decreasing function. On the other hand, for $\epsilon > 0$ and near 0, $\lambda'(\epsilon) > 0$, and $\lambda(\epsilon)$ is an increasing function. Therefore $\lambda_0 \leq \lambda(\epsilon)$ for all ϵ near 0 and the range of $\lambda(\epsilon)$ is a half open interval $[\lambda_0, \lambda_0 + \delta_1)$. This implies there are no solutions of small norm of equation (2.2) for $\lambda \in (\lambda_0 - \delta_1, \lambda_0)$. For each $\delta < 0$ the function $\lambda(\epsilon)$ restricted to $(-\epsilon_0, \delta]$ has a real analytic inverse. As before, $\lambda(\epsilon)$ restricted to $(-\epsilon_0, 0]$, has a continuous inverse $i_1(\lambda)$ and $i_1(\lambda)$ is real analytic in $(\lambda_0, \lambda_0 + \delta_1)$. Similarly, $\lambda(\epsilon)$ restricted to $[0, \epsilon_0)$ has a continuous inverse $i_2(\lambda)$ and $i_2(\lambda)$ is real analytic in $(\lambda_0, \lambda_0 + \delta_1)$. The maps $\hat{x}_1(\lambda) = x(i_1(\lambda))$ and $\hat{x}_2(\lambda) = x(i_2(\lambda))$ for $\lambda \in [\lambda_0, \lambda_0 + \delta_1)$ give us statement (iv). □

Theorem 2.4 implies the following:

Corollary 2.5. *If L and T are as in Theorem 2.1 and T is real analytic, then either λ_0 is the only eigenvalue of equation (2.2) for x of small norm, or the eigenvalues of equation (2.2) near λ_0 have at most two nontrivial solutions of small norm.*

In general, unless more is known about the operator T, λ_N of Theorem 2.4 must be computed to determine if the solutions of equation (2.2) are as in statements (i), (iii) or (iv). The next two corollaries give conditions on T which enable us to ascertain the nature of the solutions without explicitly evaluating λ_N.

Corollary 2.6. *If T is real analytic and odd (that is, $T(\lambda, -x) = -T(\lambda, x)$), then either λ_0 is the only eigenvalue near λ_0 of equation (2.2) for x of small norm, or there exist just two continuous solution functions $\hat{x}_1(\lambda)$ and $\hat{x}_2(\lambda)$ of equation (2.2) for $\lambda \in [\lambda_0, \lambda_0 + \delta)$ or $\lambda \in (\lambda_0 - \delta, \lambda_0)$ and there are no small solutions for $\lambda \in (\lambda_0 - \delta, \lambda_0]$ or $\lambda \in [\lambda_0, \lambda_0 + \delta)$.*

Proof. T is odd; therefore if (λ, x) is a solution of equation (2.2) then so is $(\lambda, -x)$. Hence there exist at least two distinct solutions for each eigenvalue. Statement (iii) of Theorem 2.4, which implies the existence of just one solution for each λ obviously does not obtain here. The solutions must therefore be as in statement (i) or (iv) of Theorem 2.4. Hence the result. □

Corollary 2.7. *Suppose T is real analytic and $T(\lambda, x) = b(\lambda)A_N(x) + R(\lambda, x)$ where $b(\lambda)$ is a real-valued analytic function, $A_N(x) = a_N(x^N)$, a_N a symmetric N-linear form, and $\|R(\lambda, x)\| = o(\|x\|^N)$ for each λ near λ_0. If in addition $[Pb(\lambda_0)A_N(u), u] \neq 0$ for $u \in N(I - \lambda_0 L)$, $u \neq 0$, then the following are true:*

(i) *If N is even there exists exactly one nontrivial continuous solution function $\hat{x}(\lambda)$ of equation (2.2) depending on λ for $\lambda \in (\lambda_0 - \delta_1, \lambda_0 + \delta_2)$.*

(ii) *If N is odd and $\lambda_0[Pb(\lambda_0)A_N(u), u] < 0$ $(\lambda_0[Pb(\lambda_0)A_N(u), u] > 0)$ for $u \in N(I - \lambda_0 L)$, $u \neq 0$, then there exist exactly two distinct nontrivial continuous solution functions. The functions, $\hat{x}_1(\lambda)$ and $\hat{x}_2(\lambda)$, depend on λ for $\lambda \in [\lambda_0, \lambda_0 + \delta)$ $(\lambda \in (\lambda_0 - \delta, \lambda_0])$ and there are no solutions of small norm of equation (2.2) for $\lambda \in (\lambda_0 - \delta, \lambda_0)$ $(\lambda \in (\lambda_0, \lambda_0 + \delta))$.*

Proof. We prove this theorem by finding λ_N of Theorem 2.4. Using the definition of $\lambda(\epsilon)$ and $\hat{\lambda}(\epsilon u_0)$ and equation (1.11) we get the equation

$$(2.8)\qquad \begin{aligned} \lambda(\epsilon) &= \hat{\lambda}(\epsilon u_0) = \{\lambda_0^{-1} + (\lambda\epsilon)^{-1}S(\lambda, \epsilon)\}^{-1} = \lambda_0\{1 + \lambda_0(\lambda\epsilon)^{-1}S(\lambda, \epsilon)\}^{-1} \\ &= \lambda_0 - \lambda_0^2(\lambda\epsilon)^{-1}S(\lambda, \epsilon) + \lambda_0 \sum_{n=2}^{\infty} (-1)^n\{\lambda_0(\lambda\epsilon)^{-1}S(\lambda, \epsilon)\}^n \end{aligned}$$

where

$$(2.9)\qquad S(\lambda, \epsilon) = [PT(\lambda, \epsilon u_0 + f(\lambda(\epsilon), \epsilon u_0)), u_0].$$

For $u \in N(I - \lambda_0 L)$,

$$\begin{aligned} PA_N(u + f(\lambda, u)) &= \sum_{i+j=N} \frac{N!}{i!j!} Pa_N(u^i, f^j(\lambda, u)) \\ &= PA_N(u) + \sum_{i+j=N;\ j\neq 0} \frac{N!}{i!j!} Pa_N(u^i, f^j(\lambda, u)). \end{aligned}$$

By Lemma 1.6 we have

$$\left\| \sum_{i+j=N;\ j\neq 0} \frac{N!}{i!j!} Pa_N(u^i, f^j(\lambda, u)) \right\| \leq \sum_{i+j=N;\ j\neq 0} c_{ij}\|u\|^i\|f(\lambda, u)\|^j = o(\|u\|^N)$$

uniformly for λ near λ_0. Therefore, by equation (2.9) $S(\lambda, \epsilon) = b(\lambda)a\epsilon^N + E(\lambda, \epsilon)$ where $a = [PA_N(u_0), u_0]$ and $|E(\lambda, \epsilon)| = o(|\epsilon|^N)$ for each λ. Since S is real analytic

$$S(\lambda, \epsilon) = b(\lambda)a\epsilon^N + \sum_{n=N+1}^{\infty} f_n(\lambda)\epsilon^n \tag{2.10}$$

where $f_n(\lambda)$ is analytic for $n = N + 1, N + 2, \cdots$. Now since $b(\lambda)$ and $\lambda(\epsilon)$ are analytic,

$$b(\lambda) = \beta_0 + \sum_{n=1}^{\infty} \beta_n(\lambda - \lambda_0)^n, \qquad \beta_0 \neq 0,$$

and

$$b(\lambda(\epsilon)) = \beta_0 + \sum_{n=1}^{\infty} \beta_n\left(\sum_{i=1}^{\infty} \lambda_i \epsilon^i\right)^n = \beta_0 + \sum_{n=1}^{\infty} b_n\epsilon^n. \tag{2.11}$$

Combining equations (2.10) and (2.11) we get

$$\hat{S}(\epsilon) = S(\lambda(\epsilon), \epsilon) = \beta_0 a\epsilon^N + \sum_{n=N+1}^{\infty} c_n\epsilon^n. \tag{2.12}$$

Since $\lambda(\epsilon)$ is real analytic and $\lambda(0) = \lambda_0$ we have

$$\lambda(\epsilon)^{-1} = \frac{1}{\lambda_0} + \sum_{n=1}^{\infty} d_n\epsilon^n. \tag{2.13}$$

From equations (2.12), (2.8) and (2.13) we get

$$\lambda(\epsilon) = \lambda_0 - \lambda_0\beta_0 a\epsilon^{N-1} + \sum_{n=N}^{\infty} e_n\epsilon^n.$$

By hypothesis, $-\lambda_0\beta_0 a = -\lambda_0[Pb(\lambda_0)A_N(u_0), u_0] \neq 0$. Hence if N is even, then $N-1$ is odd and statement (i) follows from Theorem 2.4. Similarly, if N is odd, $N-1$ is even and statement (ii) follows from Theorem 2.4. □

2.2. Let X and Y be Banach spaces. In this section we solve the bifurcation problem for the nonhomogeneous equation

$$(I - \lambda L)x - T(\lambda, x, y) = 0 \tag{2.14}$$

near $(\lambda_0, 0, 0)$ in $\mathbf{R} \times X \times Y$, where λ_0 is a simple eigenvalue of L, and L and T are as in §2.1.

For $\epsilon_0 > 0$ we let $B_{\epsilon_0} = \{(\epsilon, 0) \in \mathbf{R} \times Y \mid 0 < |\epsilon| < \epsilon_0\}$.

Theorem 2.8. *Suppose λ_0 is a simple eigenvalue of L, $I - \lambda_0 L$ is a Fredholm operator of index* 0, *and T satisfies conditions* H2–H4 *of Part* I.

(i) *There is a subset B_{ϵ_0} of $\mathbf{R} \times Y$, and an open set A containing B_{ϵ_0}, such that for each $(\epsilon, y) \in A$ equation* (2.14) *has a unique solution of the form*

$$\lambda = \lambda(\epsilon, y), \qquad x = x(\epsilon, y) = \epsilon u_0 + v(\epsilon, y), \qquad y = y,$$

where $v(\epsilon, y) \in R(I - \lambda_0 L)$ and $\lambda(\epsilon, y)$ and $v(\epsilon, y)$ are continuous. In addition,

$x(\epsilon, 0) = x(\epsilon)$ and $\lambda(\epsilon, 0) = \lambda(\epsilon)$, where $x(\epsilon)$ and $\lambda(\epsilon)$ are the unique functions of Theorem 2.1.

(ii) *If a solution $(\lambda, x, 0)$ of equation* (2.1) *is sufficiently near $(\lambda_0, 0, 0)$ then every solution of equation* (2.14) *in a small neighborhood of $(\lambda, x, 0)$ is of the form shown in statement* (i).

(iii) *If $T \in C^2$ then $\|v(\epsilon, y)\| = o(\epsilon^2 + \|y\|)$, and if $\|y\| \leq \epsilon^2$ then $|\lambda(\epsilon, y) - \lambda_0| = O(|\epsilon|)$.*

(iv) *If $T \in C^n$ then so are $\lambda(\epsilon, y)$ and $v(\epsilon, y)$.*

(v) *If T is real analytic then so are $\lambda(\epsilon, y)$ and $v(\epsilon, y)$.*

Proof. An argument similar to the one employed in the proof of Theorem 2.1 will show (λ, u, y) is a solution of equation (1.6) if and only if (λ, u, y) is a solution of equation (1.10). We will therefore find solutions of equation (1.10). Let

$$(2.15) \quad F(\lambda, (u, y)) = [u, u]\,[Lu + \lambda^{-1}PT(\lambda, u + f(\lambda, u, y), y), u]^{-1} - \lambda.$$

In the course of proving Theorem 1.7 we showed $D_1F(\lambda_0, (0, 0))$ is a linear homeomorphism. There is an open interval A_{λ_0} about λ_0 and a neighborhood A_0 of 0 in $N(I - \lambda_0 L)$ such that for $(\lambda, u) \in A_{\lambda_0} \times A_0$, $D_1F(\lambda, (u, 0))$ is a linear homeomorphism and A_0 is a subset of the domain of $\hat{\lambda}(u)$.

Now let $\hat{u} \in A_0$, $\hat{u} \neq 0$. There is a unique $\hat{\lambda} = \hat{\lambda}(\hat{u})$ such that $(\hat{\lambda}, \hat{u}, 0)$ satisfies equation (1.11). Therefore

$$[L\hat{u} + \hat{\lambda}^{-1}PT(\hat{\lambda}, \hat{u} + f(\hat{\lambda}, \hat{u}, 0)), \hat{u}] \neq 0.$$

Thus there is a neighborhood B of $(\hat{\lambda}, \hat{u}, 0)$ in which F is continuous, F is in C^n or is real analytic as T is in C^n or is real analytic, and D_1F is a linear homeomorphism. The hypotheses of the implicit function theorem are satisfied. Thus there exist an open interval $N_1(\hat{u}) \subseteq A_{\lambda_0}$ about $\hat{\lambda}$, a neighborhood $N_2(\hat{u}) \subseteq A_0$ of $\hat{u}$, a sphere $N_3(\hat{u})$ of small radius about 0 in Y, where $N_1(\hat{u}) \times N_2(\hat{u}) \times N_3(\hat{u}) \subseteq B$, and a continuous function $\hat{\lambda}(u, y)$ mapping $A(\hat{u}) = N_2(\hat{u}) \times N_3(\hat{u})$ into $N_1(\hat{u})$ such that $(\hat{\lambda}(u, y), u, y)$ satisfies equation (1.10) for all $(u, y) \in A(\hat{u})$. Moreover, if $(\lambda, u, y) \in N_1(\hat{u}) \times A(\hat{u})$ satisfies equation (1.10), then $\lambda = \hat{\lambda}(u, y)$. In addition, $\hat{\lambda}(u, v)$ is in C^n or is real analytic as T is in C^n or is real analytic.

For each $\hat{u} \in A_0$ there exist a $\hat{\lambda}$, neighborhoods $N_1(\hat{u})$ and $A(\hat{u})$, and a function $\hat{\lambda}_{\hat{u}}(u, y)$ with the above properties. We must show that if $(u, y) \in A(u_1) \cap A(u_2)$ then $\hat{\lambda}_{u_1}(u, y) = \hat{\lambda}_{u_2}(u, y)$. By our previous arguments $\hat{\lambda}_{u_1}(u, 0) = \hat{\lambda}_{u_2}(u, 0)$ (the unique solution of equation (1.11)). Let t_0 be the least upper bound for all $\hat{t}$, $0 \leq \hat{t} \leq 1$, such that for $0 \leq t \leq \hat{t}$, $\hat{\lambda}_{u_1}(u, ty) = \hat{\lambda}_{u_2}(u, ty)$. By continuity, $\hat{\lambda}_{u_1}(u, t_0 y) = \hat{\lambda}_{u_2}(u, t_0 y)$. Since D_1F evaluated at $(\hat{\lambda}_{u_1}(u, t_0 y), u, t_0 y)$ is a linear homeomorphism, an application of the implicit function theorem to equation (2.15) shows that $\hat{\lambda}_{u_1}$ and $\hat{\lambda}_{u_2}$ can be extended uniquely for $t > t_0$. Therefore $t_0 = 1$ and $\hat{\lambda}_{u_1}(u, y) = \hat{\lambda}_{u_2}(u, y)$.

Let $\hat{A} = \bigcup_{u\neq 0} A(u)$ and let A be the set of all $(\epsilon, y) \in R \times Y$ such that $(\epsilon u_0, y) \in \hat{A}$. If $(\epsilon, y) \in A$ then $(\epsilon u_0, y) \in A(\hat{u})$ for some $\hat{u}$ and we define $\lambda(\epsilon, y) = \hat{\lambda}_{\hat{u}}(\epsilon u_0, y)$. By the above argument $\lambda(\epsilon, y)$ is uniquely defined. For $(\epsilon, y) \in A$, $(\lambda(\epsilon, y), \epsilon u_0, y)$ is the unique solution of equation (1.10) and hence of (1.6). Thus, by Corollary 1.3, for each $(\epsilon, y) \in A$, $(\lambda(\epsilon, y), x(\epsilon, y), y)$ is the unique solution of equation (2.14) where

$$x(\epsilon, y) = \epsilon u_0 + v(\epsilon, y) = \epsilon u_0 + f(\lambda(\epsilon, y), \epsilon u_0, y). \tag{2.16}$$

By our construction $\lambda(\epsilon, 0) = \lambda(\epsilon)$ and $x(\epsilon, 0) = x(\epsilon)$, where $\lambda(\epsilon)$ and $x(\epsilon)$ are the solutions of equation (2.1). Thus we have proved statement (i). In addition, by Theorem 1.2 and equation (2.16) $\lambda(\epsilon, y)$ and $x(\epsilon, y)$ are in C^n or are real analytic as T is in C^n or is real analytic. Hence we have proved statements (iv) and (v).

Now we prove statement (ii). Suppose (λ, x, y) is a solution of equation (2.14) in a small neighborhood of $(\hat{\lambda}, \hat{x}, 0)$, where $(\hat{\lambda}, \hat{x}, 0)$ is in turn a solution of equation (2.1) near $(\lambda_0, 0, 0)$. By Corollary 1.3, $x = \epsilon u_0 + f(\lambda, \epsilon u_0, y)$ and $\hat{x} = \hat{\epsilon} u_0 + f(\hat{\lambda}, \hat{\epsilon} u_0, 0)$, and $(\lambda, \epsilon u_0, y)$ and $(\hat{\lambda}, \hat{\epsilon} u_0, 0)$ satisfy equation (1.6). By our previous arguments $\lambda = \hat{\lambda}_{\hat{\epsilon} u_0}(\epsilon u_0, y) = \lambda(\epsilon, y)$ and statement (ii) follows.

We proceed to prove statement (iii). Since $T(\lambda, x, y)$ is in C^2, by Theorem 1.2 $f(\lambda, u, y)$ and $PT(\lambda, u + f(\lambda, u, y), y)$ are also twice continuously differentiable. Moreover, $f(\lambda, 0, 0) = 0$, $D_2 f(\lambda, 0, 0) = 0$, $PT(\lambda, 0, 0) = 0$ and $D_2 PT(\lambda, 0, 0) = 0$ for λ near λ_0. Let $K(\lambda, u, y)$ be either $f(\lambda, u, y)$ or $PT(\lambda, u + f(\lambda, u, y), y)$. By continuity there is a neighborhood S about $(\lambda_0, 0, 0)$ such that DK and D^2K are uniformly bounded on S and if $(\lambda, u, y) \in S$ then so is (λ, tu, ty) for $0 \le t \le 1$. For $(\lambda, u, y) \in S$, using the fact that for a continuously differentiable map $\hat{T}$ of an open subset A of X into Y, if $x + ty \in A$ for $0 \le t \le 1$, then

$$\|\hat{T}(x + y) - \hat{T}(x)\| \le \sup_{0\le t\le 1} \|D\hat{T}(x + ty)(y)\| \le \|y\| \sup_{0\le t\le 1} \|D\hat{T}(x + ty)\| \tag{2.17}$$

[3, p. 155], we have

$$\begin{aligned}\|K(\lambda, u, y)\| &= \|K(\lambda, u, y) - K(\lambda, 0, 0)\| \\ &\le \|u\| \sup_{0\le t\le 1} \|D_2 K(\lambda, tu, ty)\| + \|y\| \sup_{0\le t\le 1} \|D_3 K(\lambda, tu, ty)\|.\end{aligned}$$

In addition,

$$\begin{aligned}\|D_2 K(\lambda, tu, ty)\| &= \|D_2 K(\lambda, tu, ty) - D_2 K(\lambda, 0, 0)\| \\ &\le \|u\| \sup_{0\le s\le 1} \|D_2^2 K(\lambda, su, sy)\| + \|y\| \sup_{0\le s\le 1} \|D_3 D_2 K(\lambda, su, sy)\|.\end{aligned}$$

Therefore, $\|K(\lambda, u, y)\| = O(\|u\|^2 + \|u\| \cdot \|y\| + \|y\|) = O(\|u\|^2 + \|y\|)$ independent of λ. Hence $\|v(\epsilon, y)\| = \|f(\lambda(\epsilon, y), \epsilon u_0, y)\| = O(\epsilon^2 + \|y\|)$. If $\|y\| \leq \epsilon^2$ and $E(\lambda, \epsilon) = (\lambda\epsilon)^{-1}[PT(\lambda, \epsilon u_0 + f(\lambda, \epsilon u_0, y), y), u_0]$ then by what we have just proved

$$|E(\lambda, \epsilon)| = c|\lambda\epsilon|^{-1}O(\epsilon^2 + \epsilon^2) = O(|\epsilon|).$$

Since $\lambda(\epsilon, y)$ is the unique solution of equation (1.10) for a given $(\epsilon u_0, y)$, if we let $\lambda = \lambda(\epsilon, y)$ we have

$$\lambda(\epsilon, y) = \lambda_0(1 + \lambda_0 E(\lambda, \epsilon))^{-1} = \lambda_0 - \lambda_0^2 E(\lambda, \epsilon) + \lambda_0 \sum_{n=2}^{\infty} (-1)^n(\lambda_0 E(\lambda, \epsilon))^n.$$

Thus $|\lambda(\epsilon, y) - \lambda_0| = O(|\epsilon|)$. □

Remark. Theorem 2.8 shows the existence of solutions of equation (2.14) in a neighborhood of each nontrivial solution of equation (2.1). A stronger result is true for a neighborhood of a trivial solution of equation (2.1). Differentiating the left side of equation with respect to x, and evaluating at $(\lambda, 0, 0)$, we get $I - \lambda L$. If λ_0 is an isolated element of the spectrum of L and $\lambda \neq \lambda_0$, then $I - \lambda L$ is a linear homeomorphism of X onto X. Hence by the implicit function theorem there exist a neighborhood A of $(\lambda, 0)$ in $\mathbf{R} \times Y$ and a function $h(\lambda, y)$ such that for each $(\lambda, y) \in A$, $(\lambda, h(\lambda, y), y)$ is the unique solution of equation (2.14).

Suppose equation (2.14) has the special form

$$(I - \lambda L)x - T(\lambda, x, \epsilon^2 y) = 0 \tag{2.18}$$

where L and T satisfy the hypotheses of Theorem 2.8, λ_0 is a simple eigenvalue of L, and in addition, $D_1 T(\lambda, x, y) = o(\|x\| + \|y\|^{1/2})$ uniformly for λ near λ_0.

Let y be a fixed element of Y. We prove in our next theorem the existence of a solution set of equation (2.18) branching out from $(\lambda_0, 0, 0)$ depending continuously on the real parameter ϵ.

Theorem 2.9. *Let L and T be as indicated above.*

(i) *Equation* (2.18) *has solutions of the form* $(\lambda(\epsilon), x(\epsilon), \epsilon^2 y)$ *for* ϵ *near* 0, *where* $x(\epsilon) = \epsilon u_0 + v(\epsilon)$, $v(\epsilon) \in R(I - \lambda_0 L)$. $\lambda(\epsilon)$ *and* $v(\epsilon)$ *are continuous and* $(\lambda(0), x(0), 0) = (\lambda_0, 0, 0)$.

(ii) *If T is in C^n or is real analytic then $\lambda(\epsilon)$ and $v(\epsilon)$ are in C^{n-1} or are real analytic.*

Proof. Consider solutions of the equation

$$(I - \lambda L)x - T(\lambda, x, [Px, Px]y) = 0. \tag{2.19}$$

If (λ, x) is a solution of equation (2.19) then (λ, x, ϵ^2) is a solution of equation (2.18), where $\epsilon^2 = [\epsilon u_0, \epsilon u_0] = [Px, Px]$.

Let $\hat{T}(\lambda, x) = T(\lambda, x, [Px, Px]y)$ and consider the equation $(I - \lambda L)x -$

$\hat{T}(\lambda, x) = 0$. Then since L and $\hat{T}$ satisfy the hypotheses of Theorem 2.1 the results follow. □

PART III. BIFURCATION FROM MULTIPLE EIGENVALUES

3.1. Consider the equation

$$(I - \lambda L)x - T(\lambda, x) = 0 \tag{3.1}$$

where $I - \lambda_0 L$ is a Fredholm operator of index zero. $N(I - \lambda_0 L) \cap R(I - \lambda_0 L) = \{0\}$ and T is a nonlinear map of a neighborhood of $(\lambda_0, 0)$ into X satisfying H2–H4.

Theorem 3.1. *If the above hypotheses are satisfied and* $\dim N(I - \lambda_0 L)$ *is odd then* λ_0 *is a bifurcation point of equation* (3.1).

Proof. The theorem will be proved by the following lemma. If H is a continuous map of $S_{r,n}$ into $\mathbf{R}^n$, where $\mathbf{R}^n$ is Euclidean n-space, $S_{r,n}$ is the surface of the sphere of radius r in $\mathbf{R}^n$, and n is odd, then there is some $x \in S_{r,n}$ and $\beta \in \mathbf{R}$ such that $H(x) = \beta x$. For a proof see [2, p. 51].

By this lemma for each sufficiently small positive r there is a β_r and a u_r such that $\|u_r\| = r$ and $PT(\lambda_r, u_r + f(\lambda_r, u_r)) = \beta_r u_r$ where $\lambda_r = \hat{\lambda}(u_r)$. By Corollary 1.9 $(\lambda_r, u_r + f(\lambda_r, u_r))$ is a solution of equation (3.1). Therefore, by the continuity of $\hat{\lambda}(u)$, λ_0 is a point of bifurcation. □

3.2. When $\dim N(I - \lambda_0 L)$ is even solving the bifurcation problem of equation (3.1) becomes more difficult. There is a simple example in which L is a self-adjoint operator in a Hilbert space, $\dim N(I - \lambda_0 L) = 2$, L and T satisfy H1–H4 and T depends linearly on λ, yet there are no nontrivial solutions of equation (3.1) near $(\lambda_0, 0)$ (see [1, p. 121]). Thus in general when $\dim N(I - \lambda_0 L)$ is even additional restrictions must be imposed on T to guarantee the existence of solutions of small norm for λ near λ_0.

Let us assume $\dim N(I - \lambda_0 L) = 2$.

Theorem 3.2. *Suppose there exists a normalized basis* $\{u_1, u_2\}$ *of* $N(I - \lambda_0 L)$ *such that* $[PT(\lambda, bu_2 + f(\lambda, bu_2)), u_1] \geq 0$ *for small* $|b|$ *and all* λ *near* λ_0, *where* $[\ ,\ ]$ *is defined as in* §1.3. *Then* λ_0 *is a bifurcation point of equation* (3.1) *with respect to* 0.

Proof. PT, $f(\lambda, u)$ and $\hat{\lambda}(u)$ are continuous in $N(I - \lambda_0 L)$ with norm $\|\ \|_H = [\ ,\ ]^{1/2}$. For $x, y \in N(I - \lambda_0 L)$, $x = \beta y$ if and only if $[x, z] = 0$, $z \in \{y\}^{\perp}$.

For convenience we denote $au_1 + bu_2$ by (a, b). With this convention

$$PT(\lambda, (a, b) + f(\lambda, a, b)) = (T_1(\lambda, a, b), T_2(\lambda, a, b)),$$

where $T_1 = [PT, u_1]$ and $T_2 = [PT, u_2]$.

For each $(a, b) \neq (0, 0)$, $(b, -a)$ is a basis of $\{(a, b)\}^{\perp}$. Therefore, $PT(\lambda, (a, b) + f(\lambda, a, b)) = \beta(a, b)$ if and only if

$$bT_1(\lambda, a, b) - aT_2(\lambda, a, b) = 0. \tag{3.2}$$

Choose $c_0 > 0$ so that $\hat{\lambda}(u)$ is defined for all u, $\|u\|_H \leq c_0$. Let $0 < c < c_0$. By hypothesis for all λ near λ_0, $cT_1(\lambda, 0, c) \geq 0$ and $-cT_1(\lambda, 0, -c) \leq 0$. The function $H(a, b) = bT_1(\hat{\lambda}(a, b), a, b) - aT_2(\hat{\lambda}(a, b), a, b)$ maps the connected set $\{u \in N(I - \lambda_0 L) \mid \|u\|_H = c\}$ into $\mathbf{R}$. We have seen that $H(0, c) \geq 0$, and $H(0, -c) \leq 0$. Therefore, there is a point $(\bar{a}, \bar{b})$ of norm c such that $H(\bar{a}, \bar{b}) = 0$. Since $(\hat{\lambda}(\bar{a}, \bar{b}), (\bar{a}, \bar{b}))$ satisfies equation (3.2) we obtain the equation $PT(\bar{\lambda}, \bar{u} + f(\bar{\lambda}, \bar{u})) = \beta\bar{u}$ where $\bar{u} = (\bar{a}, \bar{b})$, $\|\bar{u}\|_H = c$ and $\bar{\lambda} = \hat{\lambda}(\bar{u})$. By Corollary 1.9 $(\bar{\lambda}, \bar{u} + f(\bar{\lambda}, \bar{u}))$ is a solution of equation (3.1). For each small c, there is a solution $(\lambda_c, u_c + f(\lambda_c, u_c))$ of equation (3.1). $\|u_c\|_H = c$ and $\lambda_c = \hat{\lambda}(u_c)$. Therefore, by the continuity of $\hat{\lambda}(u)$, λ_0 is a point of bifurcation of equation (3.1).

PART IV. COUNTEREXAMPLES

4.1. In this section we give three examples to demonstrate that when some of our hypotheses are removed many of the results of §2.1 are not valid. In all of our examples, T depends linearly on λ, $\dim N(I - \lambda_0 L) = 1$, $T \not\equiv 0$ and T satisfies the hypotheses of Theorem 2.1.

Example 1. If λ_0 is not a simple eigenvalue of L then λ_0 may fail to be a bifurcation point of equation (2.2) with respect to 0.

Consider the system of equations in $\mathbf{R}$:

$$x - \lambda(x + y + y^3) = 0, \tag{4.1}$$

$$y - \lambda(y - x^3) = 0. \tag{4.2}$$

Here T is an odd operator defined by $T(x, y) = (y^3, -x^3)$, $\lambda_0 = 1$ is an eigenvalue of L. A computation will show $\dim N(I - \lambda_0 L) = 1$ but λ_0 is not simple. In addition, T satisfies the hypotheses of Theorem 2.1 and is real analytic. This system has no solutions for λ near $\lambda_0 = 1$ other than $x = y = 0$. To see this, suppose (λ, x, y) is a solution. Multiplying equation (4.1) by y and equation (4.2) by x, and subtracting we get $\lambda(y^2 + y^4 + x^4) = 0$. Therefore, $x = y = 0$.

Example 2. In this example $L + T$ is a gradient operator, T is real analytic, λ_0 is a simple eigenvalue and $I - \lambda_0 L$ is a Fredholm operator of index 0. However, though λ_0 is a bifurcation point, λ_0 is the only eigenvalue of the nonlinear problem for x of small norm.

Consider the equations in $\mathbf{R}$,

$$x - \lambda(-2x - x^3) = 0, \tag{4.3}$$

$$y - \lambda y = 0. \tag{4.4}$$

L and T are as indicated above and $\lambda_0 = 1$ is a simple eigenvalue of L. Equation (4.3) has no solutions other than $x = 0$ for λ near 1. Suppose $x \neq 0$ is a solution of equation (4.3). Then $(1 + 2\lambda)x + \lambda x^3 = 0$. Dividing by x and then by

λ we get $\lambda^{-1}(1+2\lambda)+x^2=0$. But this equation has no real solution since $\lambda^{-1}(1+2\lambda)>0$. On the other hand, equation (4.4) has no solution other than $y=0$ if $\lambda \neq 1$. λ_0 is thus the only eigenvalue for small $|x|$.

Example 3. Here T is n times continuously differentiable, L and T satisfy the hypotheses of Theorem 2.1, and λ_0 is not the only eigenvalue, yet there is one eigenvalue with infinitely many (nonlinear) eigenfunctions x. Moreover, for each integer N, there exists a sequence of eigenvalues, different from λ_0, such that each eigenvalue has at least N eigenvectors and the eigenvalues and eigenvectors converge to λ_0 and 0 respectively.

Consider the equation

$$x - \lambda(x + x^m \sin(x^{-1})) = 0 \tag{4.5}$$

for $x \in \mathbf{R}$. $x^m \sin(x^{-1})$ is n times continuously differentiable for $m = 2n+1$. Clearly the hypotheses of Theorem 2.1 are satisfied and $\lambda_0 = 1$ is a simple eigenvalue of the linearized part. Dividing (4.5) by x we get

$$(1-\lambda) + x^{m-1}\sin(x^{-1}) = 0. \tag{4.6}$$

Since $x^{m-1}\sin(x^{-1}) \not\equiv 0$, λ_0 is not the only eigenvalue. For $\lambda = \lambda_0$ there correspond infinitely many solutions of small norm.

Let N be an integer. Then since $\sin(x^{-1})$ oscillates about 0, we can find a sequence $\{\lambda_n\}$, $\lambda_n \to \lambda_0$, such that for each λ_n there correspond N distinct values $x_{n_1}, \cdots, x_{n_N}$ and each (λ_n, x_{n_i}) satisfies equation (4.6) and $x_{n_i} \to 0$.

Since solutions of equation (4.6) are also solutions of equation (4.5) we have shown the existence of the sequence described above.

PART V. APPENDIX

We will now give the proofs of those theorems cited in previous sections without proof.

Theorem 5.1. *Let L be a bounded linear operator mapping a Banach space X into itself. Suppose $N(L)$ is finite-dimensional. Then $X = N(L) \oplus R(L)$ if and only if L is a Fredholm operator of index zero and $N(L) \cap R(L) = \{0\}$.*

Proof. We denote the space of all bounded linear functionals on X by X', the adjoint of L by L' and for any subspace N, $N^\perp$ is the set of annihilators of N.

Suppose $X = N(L) \oplus R(L)$. Then $R(L)$ is closed and $R(L)^\perp = N(L')$. Let $m = \dim N(L)$. Then the dimension of the dual space $N(L)'$ of $N(L)$ must also be be m. But $R(L)^\perp$ is equivalent to $N(L)'$. Hence $\dim N(L') = m$ and L is a Fredholm operator of index 0.

Suppose now L is Fredholm and $\operatorname{ind}(L) = 0$. Let $m = \dim N(L) = \dim N(L')$. $N(L) \oplus R(L)$ is closed. Suppose $X \neq N(L) \oplus R(L)$. Then there is a nonzero vector $x_0 \in X$ such that $x_0 \notin N(L) \oplus R(L)$. Let $x_1, x_2, \cdots, x_m$ be a basis of $N(L)$, By the Hahn-Banach theorem there exist $m+1$ bounded linear functionals $y_0, y_1, \cdots, y_m$ such that for $0 \le i, j \le m$; $[x_i, y_j] = 0$ for $i \neq j$, $[x_i, y_i] = 1$ and $[x, y_i] = 0$ for $x \in R(L)$. The y_i are linearly independent elements of $R(L)^\perp$ and $R(L)^\perp = N(L')$. However, $\dim N(L') = m$ so there cannot be $m+1$ linearly independent vectors in $N(L')$. Therefore $X = N(L) \oplus R(L)$. □

We now prove Lemma 1.5.

To show statement (i), choose $\epsilon > 0$. By hypothesis there is a neighborhood A of $(x_0, 0)$ such that if $(x, y) \in A$ then $(x, ty) \in A$ for $0 \le t \le 1$ and

$$\sup_{(x,y)\in A} \|D_2T(x, y)\| \le c < \infty.$$

Let $\delta = \epsilon/c$. Then for any $(x, y) \in A$ with $\|y\| \le \delta$ we have by inequality (2.17)

$$\|T(x, y)\| = \|T(x, y) - T(x, 0)\| \le \|y\| \sup_{0 \le t \le 1} \|D_2T(x, ty)\|$$

$$\le \|y\|c \le \delta c \le (\epsilon/c)c = \epsilon.$$

To show statement (ii), let $A = N_1 \times N_2$ where N_1 is the neighborhood of x_0 specified in the hypothesis of (ii); N_2 is a neighborhood of $y = 0$ such that if $y \in N_2$, then $ty \in N_2$ for $0 \le t \le 1$; and $N_1 \times N_2$ is in the domain of T. Choose $\epsilon > 0$. There exists a $\delta > 0$ such that $\|D_2T(x, y)\| < \epsilon$ for $(x, y) \in A$ with $\|y\| \le \delta$. Hence for $(x, y) \in A$ with $\|y\| \le \delta$,

$$\|T(x, y)\| = \|T(x, y) - T(x, 0)\| \le \|y\| \sup_{0 \le t \le 1} \|D_2T(x, ty)\| \le \|y\|\epsilon.$$

Therefore, $\|T(x, y)\| = o(\|y\|)$. □

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